Phase coexistence in a forecasting game

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In everyday life, our choices are either based on personal experience or on information provided by our neighborhood. In the latter case, our choices conform to those of the majority of the agents in our neighborhood. Such herding behavior may be very efficient in aggregating disperse private information, thereby revealing the optimal choice. However if the majority relies on herding, this mechanism may dramatically fail to aggregate correctly the information, causing the majority adopting the wrong choice. We address these issues in a simple model of interacting agents who aim at giving a correct forecast of a public variable, either seeking private information or resorting to herding. As the fraction of herders increases, the model features a phase transition beyond which a state where most agents make the correct forecast coexists with one where most of them are wrong. Simple strategic considerations suggest that indeed such a system of agents self-organizes deep in the coexistence region. There, agents tend to agree much more among themselves than with what they aim at forecasting, as found in recent empirical studies.

Information affects in many subtle ways socio-economic behavior, giving rise to non-trivial collective phenomena. For example, a key function of markets is that of aggregating the information scattered among traders into prices. However, if traders rely on the information conveyed by prices, this same mechanism may lead to self-sustaining speculative bubbles. Likewise, we deduce the worth of a restaurant or the importance of a research subject from its crowdedness or popularity. However, popularity can consecrate even totally random choices [1].

Collective herding phenomena in general pose quite interesting problems in statistical physics. To name a few examples, anomalous fluctuations in financial markets and opinion dynamics have been related to percolation theory based models of herding [2, 3, 4, 5] and social changes have been shown to follow patterns which are well explained by the theory of random field Ising models [6]. It is natural to expect herding behavior in cases when it is convenient for the individuals to follow the herd. For example, when the majority is buying in the stock market, prices goes up, hence buying becomes the right thing to do (at least in the short run). If a technology (e.g. fax machine) is widely adopted, it becomes more convenient to adopt it. Herding takes place even in cases where agents' behavior does not influence the outcome, if agents try to infer information about the optimal choice from the actions of others. Ref. [1] discussed how these considerations are important for issues ranging from the prevalence of crime, marketing, fads and fashions to the onset of protests such as that leading to the collapse of the East German regime. Ref. [7] remarks that herding might explain why financial forecasters tend to make very similar predictions – whose diversity is much smaller than the prediction's error.

From the theoretical side, the onset of herding and the resulting failure of information aggregation has been shown to occur in models of *information cascades* [1]. The prototype example is that of a set of agents who have to chose one of two restaurants on the basis of some private noisy information. If each of them chooses simultaneously according to his/her private signal, the majority will chose the best restaurant. However if agents take choices one after the other and each can observe what others have chosen before, the result changes dramatically. From a certain point onward, the behavior of the majority provides more information than that of private signal, hence it is optimal for agents to follow the majority, disregarding their private signal. As a result, choices disclose no further information and there is a sizeable probability that they all enter the worse restaurant.

In this letter, we show that information herding can bring to non trivial collective phenomena even when agents observe a finite number of peers and act in no particular order. In particular, a population of selfish agents fails to correctly aggregate information because herding brings the system into a coexistence region, where the vast majority of agents "agrees" on the same forecast, not necessarily the right one. A statistical mechanics approach gives a detailed account of the results in terms of a zero temperature Ising model with asymmetric interaction. These insights extend to the case where agents have to forecast a variable in a continuous interval. Again we find a spinodal point beyond which forecasts tend to cluster, as observed in Ref. [7].

Let us consider a population of agents who have to forecast the value of a binary event $E \in \{\pm 1\}$. Each agent $i = 1, \ldots, N$ faces the choice of either looking for information or herding. We shall denote by I and H, respectively, these two strategies, as well as the set of agents who follow them. In the former case agent $i \in I$ receives a signal f_i which is draw with $P\{f_i = E\} = p = 1 - P\{f_i = -E\}$, independently for each agent $i \in I$. We assume that the signal f_i is informative about E, i.e. $p > 1/2$. In the case of strategy H, agent i forms a sample group G_i by picking an odd number K of other agents at random, observes their forecasts f_j and sets his/her forecast to that of the majority of agents $j \in G_i$. Notice first that $j \in G_i$ – i.e. i observing j – does not imply that $i \in G_j$ – i.e. that j observes i. Secondly, the forecast of

i may depend on the forecast of other agents who are themselves herding. Hence we assume that forecasts are formed by an iterative process

$$f_i^{(\tau+1)} = \text{sign} \sum_{j \in G_i} f_j^{(\tau)}, \qquad \forall i \in H \tag{1}$$

where $f_j^{(0)}$ is drawn at random with $P\{f_j^{(0)} = \pm 1\} = 1/2$ for all $j \in H$ and $f_j^{(\tau)} = f_j$ stays constant for all $j \in I$. We denote simply by f_j the fixed point value of the forecast resulting from this process. Both strategies imply a cost, which for simplicity we assume to be the same: either agents invest in information seeking or in forming a sample group. In other words, agents have access to either type of information but not both. We assume that the goal of agents is that of reaching a correct forecast, i.e. that the payoff of agent i is the probability $P\{f_i = E\}$ that his/her forecast is right. By definition, $P\{f_i = E | i \in I\} = p$, whereas the probability that an herding agent forecasts the correct outcome is

$$q = \frac{1}{\eta N} \sum_{i \in H} \delta_{f_i, E}. \tag{2}$$

where we introduced the fraction η of agents who follow the H strategy.

Let us first focus on the case where the fraction η of agents $i \in H$ is fixed and then move to the case where this is fixed by agents' optimizing behavior. The inset of Fig. 1 shows the behavior of q as a function of η in typical numerical simulations. The average $\langle q \rangle$ of q over different realizations is reported in Fig. 1. When η is small, herding is quite efficient and it yields more accurate predictions than information seeking ($\langle q \rangle > p$). Actually the probability $\langle q \rangle$ that H-players end up with the correct forecast increases with η up to a maximum. This is because herders use the information of other herders who have themselves a higher performance than private information forecasters. However beyond a certain point, outcomes with a value $q < p$ start to appear, coexisting with outcomes with $q \approx 1$. Consequently the average $\langle q \rangle$ starts decreasing. The low q state becomes more and more probable as η increases, and for η close to one we find $\langle q \rangle < p$.

In order to shed light on the above results, let us notice that the probability of a randomly drawn agent to give the right forecast is

$$P\{f_i = E\} \equiv \pi = (1-\eta)p + \eta q. \tag{3}$$

In order to derive an equation for q we observe that a herding agent adopts the point of view of the majority of his K randomly drawn agents, i.e.

$$q = \Sigma_K(\pi) \equiv \sum_{g=(K+1)/2}^{K} \binom{K}{g} \pi^g (1-\pi)^{K-g} \tag{4}$$

These are two self consistent equations for q. For a given value of p, the solution is unique for $\eta < \eta_c(p, K)$ whereas for $\eta > \eta_c(p, K)$, as shown in Fig. 1, we find three solutions, which we denote by $q_+ > q_u > q_-$. The critical point η_c increases with p and with K.

A direct calculation shows that the average number of fixed points of Eqs. (1) is dominated by configurations $\{f_i\}$ for which q satisfies Eqs. (3,4). Interestingly, we find that the average number of fixed points $\mathcal{N} \simeq (K^K e^{-K}/K!)^{\eta N} [p(1-p)]^{-(1-\eta)N}$ is the same on all the solutions. Linear stability of the dynamics (1), however, shows that the fixed points $q_\pm$ are stable whereas the one at q_u is unstable. The unstable solution q_u separates the basin of attraction of the fixed points $q_\pm$. This allows us to estimate the probability p_- that the system converges to the fixed point q_-, which is the probability that the initial value of $q(0)$ falls below q_u. Given that variables $f_i^{(0)}$ are assigned a random sign for $i \in H$, $q^{(0)}$ is well approximated by a gaussian variable of mean zero and variance $1/(\eta N)$. Hence

$$p_- \equiv P\{q(0) < q_u\} \cong \frac{1}{2} \text{erfc} \left(\sqrt{\eta N/2}(1 - 2q_u) \right). \tag{5}$$

The expected value of q is then given by

$$\langle q \rangle = p_- q_- + (1 - p_-) q_+. \tag{6}$$

Fig. 1 shows that Eq. (6) agrees very well with numerical simulations for large N. The discrepancy for small N comes from the fact that indeed the dynamics of $q^{(\tau)}$ is subject to a noise term of order $1/\sqrt{N}$ which causes transitions across q_u in the early stages of the dynamics for small N. It is easy to show that, for $\eta \approx 1$,

$$q_u \simeq \frac{1}{2} - \frac{(p-1/2)k!!}{k!! - (k-1)!!}(1-\eta) + O(1-\eta)^2 \tag{7}$$

which shows that there is a window of size $1/\sqrt{N}$ close to $\eta = 1$ where p_- is sizeable. As a consequence, the fall of q in this region gets steeper and steeper as N increases.

This consideration is important is we analyze the behavior of selfish agents following game theory [8]. We assume for simplicity that agents aim at reaching a correct forecast, i.e. that their payoff is the probability that $f_i = E$. As long as $\langle q \rangle > p$ agents will find it more convenient to switch from the I to H strategy. Hence, the fraction η of herders increases when $\langle q \rangle > p$. The contrary is true when $\langle q \rangle < p$ and hence we expect that the population will self-organize to a state η^*, such that no agent has incentive to change strategy, i.e. where $\langle q \rangle = p$. Such a state is called a Nash equilibrium [8]. Its standard interpretation as the equilibrium of forward looking rational agents, who correctly anticipate the behavior of others, given the rules of the game, and respond optimally, requires agents who are able to solve a rather complex statistical mechanical problem. We will however show below that adaptive agents with limited rationality can "learn" to converge to such a Nash equilibrium.

In the Nash equilibrium all but a fraction of order $1 - \eta^* \sim \sqrt{N}$ of agents takes the H strategy. In addition,

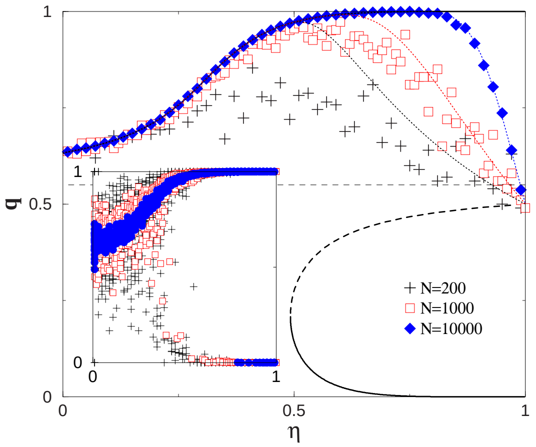

FIG. 1: The average success q of herding agents is shown, for simulations (symbols) and for the analytical solution (dotted lines) as a function of the herding probability η for $K = 11$, $p = 0.55$ and $N = 200$ (+) 10^3 (□) and 10^4 (⋄) agents. The stable solutions $q_\pm$ are shown as full lines whereas the unstable one q_u is shown as a dashed line. Inset: individual realizations of q for the same systems above.

because in this region $q_+ \cong 1$ and $q_- \cong 0$, we have $p_- \cong 1 - p$. This means that the whole population adopts the wrong forecast with probability $1-p$, as if it were a single individual forecasting on the basis of private information. Such a spectacular event is similar to the outcome of information cascades [1], but it takes place in a quite different setting.

Does this scenario changes when we introduce heterogeneity in agents' characteristics? Let us first consider the case where agent i, when using strategy H, can observe K_i peers. Naïvely one would expect that agents with larger K_i receive more precise information and hence should prefer the H strategy. However, because at the Nash equilibrium almost every agent is making the same prediction, either right or wrong, a larger "window" K_i does not help. The case where agents have different individual forecasting abilities, i.e. when p_i depends on i, is a bit more complex. It is reasonable to assume that "expert" agents with $p_i > \langle q \rangle$ will seek private information whereas those with $p_i < \langle q \rangle$ will herd. Again q is given by Eqs. (3,4) with

$$\eta = \int_0^{\langle q \rangle} dp\ \phi(p), \qquad (1-\eta)p = \int_{\langle q \rangle}^1 dp\ p\ \phi(p) \qquad (8)$$

where $\phi(p)$ is the distribution of p_i. It is easy to show that a solution of Eqs. (3,4,8) with $q = \langle q \rangle$, i.e. where η and p do not fall in the coexistence region is not possible. Indeed the only solution of $\Sigma_K[q\int^q dp\phi(p) + \int_q^1 dpp\phi(p)] = q$ is at $q = 1$, which implies $\eta = 1$. The solution then lies in the coexistence region, where Eqs. (3,4) have three solutions, and it is found computing $\langle q \rangle$ as before from Eqs. (5,6) as a function of η and p, and then using Eq. (8) to compute η and p self-consistently. The results are shown in Fig. 2 for $\phi(p) = \beta 2^\beta (1-p)^{\beta-1}$, $p \in [1/2, 1]$. When β is large, there is small heterogeneity and we are back to the case $p_i = p$: Almost all agents follow the H strategy $q \approx 1$ and the probability of a wrong forecast $p_- \simeq 1/2$ is large. As β decreases, the number of "experts", i.e. agents with $p_i > q$ increases, and correspondingly also the performance of the population as a whole improves (i.e. q increases and p_- decreases). In this region, asymptotic analysis shows that the fraction of "experts" $1 - \eta \sim \sqrt{\log N/N}$.

The analytical results were tested against numerical simulations of adaptive agents who repeatedly play the game and learn, in the course of time, about their optimal choice. In order to do this, agents compute the cumulative payoff for both strategies and adopt the strategy with the largest score [9]. As expected, we find that in each run there is a value q such that all agents with $p_i > q$ play the I strategy whereas those with $p_i < q$ herd. Again some deviations occur for small N but the agreement improves as N increases. This shows that the type of equilibria we discuss are "learnable" by a population of not extremely sophisticated agents. It is well known that the type of reinforcement learning dynamics discussed above has close analogies with evolutionary dynamics [10]. Hence the scenario we discussed above, might as well describe social norms which are the result of evolutionary processes.

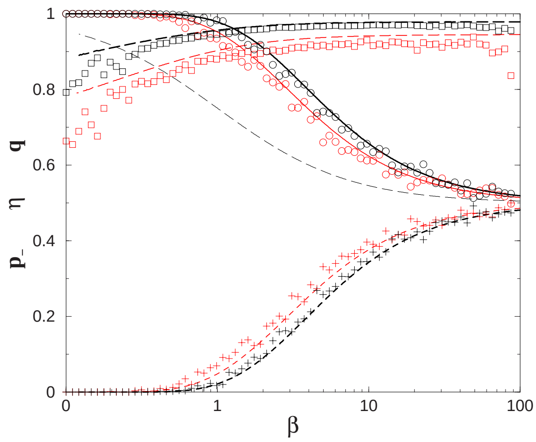

FIG. 2: Analytical results (lines) compared to numerical simulations (symbols) for systems of $N = 100$ and 800 agents with heterogeneous forecasting ability p_i drawn from the distribution $\phi(p) = \beta 2^\beta (1-p)^{\beta-1}$. The average success q (full line and ∘), the fraction η of herding agents (long dashed line and □) and the probability p_- that the majority forecasts the wrong outcome (short dashed line and +), as a function of β. For comparison, the thin dashed line shows the average success of agents with no herding ($\eta = 0$).

The insights of the discrete model hold also when agents have to forecast a continuous variable E. In order to show this, we adopt an asymmetric version of the continuous opinion model of Ref. [4], where a population of N agents submits forecasts $\{f_i\}$ of a continuous event $E \in [0, 1]$. Again, forecasters may either seek private information

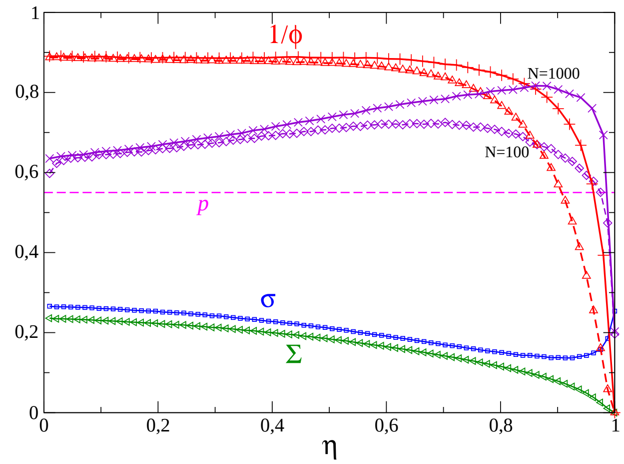

FIG. 3: Continuous forecasting model for $K = 11, d = \mu = 0.5, \varepsilon = 0.1$. The inverse herding parameter ϕ^{-1} is only of the order of 0.1 for a strong herding regime near the Nash equilibrium $\eta \approx 0.98$. The dispersion σ and the error Σ are only shown for $N = 100$. Note that η_{Nash} increases with N whereas ϕ_{Nash} decreases.

(strategy I) or herd (strategy H). All I agents receive a signal $f_i \in [0,1]$ which, with probability p is "correct", i.e. is randomly drawn from the interval $[E-\epsilon, E+\epsilon]$, and with probability $1-p$ is uniformly distributed in $[0,1]$. If instead $i \in H$, we draw at random sample groups G_i of K agents and assign initial random values $f_i^{(0)} \in [0,1]$ to herding agents. Then we iterate the dynamics over agents j of the the group G_i

$$f_i^{(\tau+1)} = f_i^{(\tau)} + \mu(f_j^{(\tau)} - f_i^{(\tau)})\;\theta\left(d - |f_j^{(\tau)} - f_i^{(\tau)}|\right)$$

until $|f_i^{(\tau+1)} - f_i^{(\tau)}| < \epsilon$. We denote simply by f_i the limit value of $f_i^{(\tau)}$ in this process. Note that agent i is influenced by $j \in G_i$ only if their opinion are not too far, i.e. if $|f_j^{(\tau)} - f_i^{(\tau)}| < d$. Forecasts are considered to be correct if $|f_i - E| < \epsilon$.

As in Ref. [7], we introduce the forecast error $\Sigma = \sqrt{\langle(\bar{f} - E)^2\rangle}$ and the forecast dispersion $\sigma = \sqrt{\langle\overline{(f_i - \bar{f})^2}\rangle}$ where $\bar{\cdot}$ denotes the average over agents whereas the average $\langle\ldots\rangle$ is taken over different realizations of the process. The ratio $\phi = \Sigma/\sigma$ called the empirical herding coefficient, is a measure of herding as explained in ref. [7].

Fig. 3 shows the results of numerical simulations of the model as a function of the fraction η of herders, for a typical choice of the parameters. As in the discrete model, we find that for small values of η the probability $q = P\{|f_i - E| < \epsilon | i \in H\}$ of a correct forecast for herders is larger than that of information seeking agents (p) and it increases because herding agents aggregate the information of other agents who are also herding. Upon increasing η further, q reaches a maximum and then it decreases as the information entering in the system diminishes. In this region, we find coexistence of a state where the vast majority of agents are right with a state where almost all of them are wrong. The Nash equilibrium, where both strategy are equally successful ($\langle q\rangle = p$), is precisely in this region and the herding coefficient attains values $\phi \simeq 5 \div 10$, which are comparable to those found in Ref. [7] on a survey of earning forecasters of US, EU, UK and JP stocks during the period 1987-2004. The fact that analysts agree with each other five to ten times more than with the actual result, was claimed to be related to herding effects Ref. [7], a conclusion fully supported by our results. Furthermore, as in the discrete model, the Nash equilibrium moves towards $\eta = 1$ as N increases, thus making herd behavior more pronounced. Our simple model then suggests that ϕ should take larger values in markets with a larger number of forecasters. This might explain some of the differences found in Ref. [7] across markets.

In conclusion, we introduced a simple model capturing the tension between private information seeking and exploiting information gathered by others (herding) in a population. When few agents herd, information aggregation is very efficient. This makes herding the choice taken by nearly the whole population, thus setting the system deep in a "coexistence" region where the population as a whole adopts either the right or the wrong forecast. This scenario is rather robust and applies both to a discrete and a continuum model and it compares well with empirical findings [7]. The model and the statistical mechanics analysis can serve as a basis to address a wide range of related issues.

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[1] S. Bikhchandani, D. Hirshleifer and I. Welch, J. Pol. Econ. **100** (1992).

[2] R. Cont and J. Bouchaud, Macroeconomic Dynamics **4**, 170 (2000).

[3] D. Stauffer, Adv. Complex Syst. **4** (2001).

[4] G. Weisbuch and *alter*, Complexity **7**, 55 (2002).

[5] V. Eguíluz and M. Zimmermann, Phys. Rev. Lett. **85**, 5659 (2003).

[6] Q. Michard and J.-P. Bouchaud, cond-mat **0504079** (2005).

[7] O. Guedj and J.-P. Bouchaud, cond-mat **0410079** (2004).

[8] F. Vega-Redondo, *Economics and the theory of games* (Cambridge Univ. Press, 2004).

[9] D. Challet, M. Marsili and Y.-C. Zhang, *The Minority Game* (Oxford Univ. Press, 2004).

[10] T. Borgers and R. Sarin, J. Econ. Th. **77** (1997).